Journal of Engineering Physics and Thermophysics, Vol. 76, No. 1, 2003

HEAT TRANSFER FROM A JET FLOWING INTO A CLOSED CAVITY

G. V. Tkachenko and B. A. Uryukov

UDC 532.526

The exact solution of the equations of incompressible-fluid dynamics which is interpreted as jet flow in a closed cavity is found. The boundary-layer equations on the cavity walls are obtained and the problem of heat exchange between the jet and the walls is solved. It is shown that as the jet source moves away from the cavity bottom, the Nusselt number decreases exponentially.

The Whittaker integral [1, 2] allows determination of the solution for an axisymmetric potential incompressible-fluid flow using the known velocity distribution on the axis. If $f_0(z)$ is the axial-velocity distribution, then the solution has the following form:

the longitudinal velocity is

$$V_z = f_0(z) + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} r^{2n} f_0^{(2n)}(z) , \tag{1}$$

the radial velocity is

$$V_r = \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{2^{2n}(n!)^2} r^{2n-1} f_0^{(2n-1)}(z) , \tag{2}$$

the stream function is

$$\psi = -\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{2^{2n}(n!)^2} r^{2n} f_0^{(2n-2)}(z) . \tag{3}$$

If $f_0(z)$ is the linear exponential function

$$f_0(z) = A + B \exp(az) , \tag{4}$$

then the solutions (1)–(3) can be represented as a sum of the products exp (az) and the functions of the radius r. For example, the stream function is

$$\psi = A \frac{r^2}{2} + B \frac{\exp(az)}{a^2} P(ar) .$$

The determinations of the velocities performed in terms of the stream function give

$$V_z = \frac{1}{r} \frac{\partial \psi}{\partial r} = A + B \frac{\exp(az)}{a} \frac{P'}{r} , \quad V_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} = -B \frac{\exp(az)}{a} \frac{P}{r} ,$$

Institute of Problems of Materials Science, National Academy of Sciences of Ukraine, Kiev, Ukraine; email: cosmos@ipms.kiev.ua. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 76, No. 1, pp. 65–70, January–February, 2003. Original article submitted May 31, 2002.

1062-0125/03/7601-0076$25.00

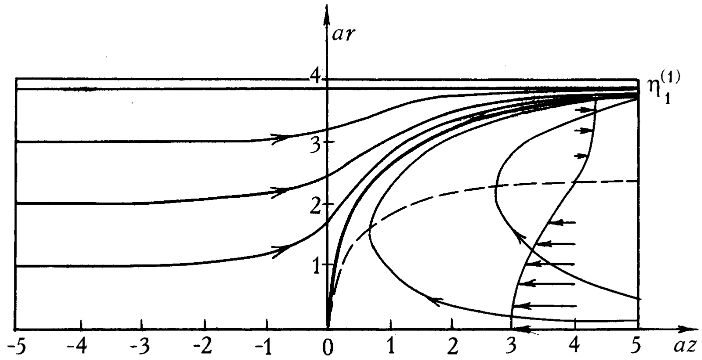


Fig. 1. Pattern of streamlines in the flow.

where the prime denotes differentiation with respect to the argument.

From the condition of flow potentiality

$$\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = 0$$

we find the equation for defining the function P

$$P''\eta - P' + P\eta = 0\,, \quad \eta = ar\,,$$

whose solution is expressed by the Bessel function

$$P = \eta J_1(\eta)\,.$$

Now we consider the flow resulting on condition that the velocity at a certain axial point vanishes. Without loss of generality, this point can be taken for $z = 0$. Then, from Eq. (4) it follows that $B = -A$.

Thus, the stream function is

$$\psi = Ar\left(\frac{r}{2} - \frac{\exp(az)}{a} J_1(ar)\right),$$

the longitudinal and radial velocities are as follows:

$$V_z = A\,[1 - \exp(az)\,J_0(ar)]\,, \quad V_r = A\exp(az)\,J_1(ar)\,. \tag{5}$$

This solution has the form shown in Fig. 1.

Flow inside the cavity (right of Fig. 1) is one physical interpretation. The equation for the cavity contour is

$$\exp(az) = \frac{ar}{2J_1(ar)}\,. \tag{6}$$

The constant A is determined from the condition of assignment of the velocity ($V_z = -V_0$) on the axis at a certain distance h from the stagnation point. Thus, in Fig. 1 we showed the velocity profile V_z related to the axial velocity at the point $az = ah = 4$. The constant a can be found by assignment of the limiting cavity radius R_c:

$$a = \frac{\eta_1^{(1)}}{R_c}\,. \tag{7}$$

In technological processes, there are cases where the jet of a liquid or a gas flows into an axisymmetric cavity with a contour of the type (6), for example, in plasma or fire boring, drilling, and cutting. In these processes, it is

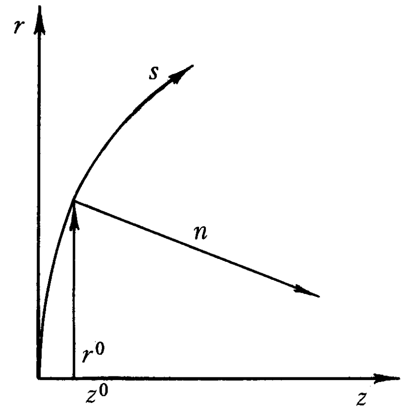


Fig. 2. Curvilinear orthogonal coordinate system (in the plane φ = const).

necessary to know the conditions of heat transfer from the jet to the cavity walls on which the boundary layer grows. Since the walls are curvilinear, the boundary-layer equations can conveniently be analyzed provided that they are written in the corresponding curvilinear coordinates. As coordinates of this type we take the coordinate system (n, s, φ) [3] shown in Fig. 2. The coordinate s is reckoned from the jet axis and is directed along the contour, n is the straight line directed normal to s inside the cavity, and φ is the angle (it is the same as in the initial system). In principle, this system is ambiguous, since the neighboring normals n intersect at a distance which is equal to the radius of curvature of the contour. However, we consider a region extended insignificantly along the n axis and considerably smaller than the radius of curvature at all the points. Therefore, within the limits of the region (of the boundary layer), none of the normals intersects another.

The quadratic form in the selected coordinate system appears as

$$dl^2 = \left(1 + \frac{n}{R(s)}\right)^2 ds^2 + dn^2 + \left(r^0 + n\frac{dz^0(s)}{ds}\right)^2 d\varphi^2 ,$$

where $R(s)$ is the radius of curvature of the coordinate axis s, i.e., of the cavity contour. Whence we determine the Lamé coefficients

$$L_s = 1 + \frac{n}{R(s)}, \quad L_n = 1, \quad L_\varphi = r^0 + n\frac{dz^0(s)}{ds} .$$

Using the formulas presented in [4], we write the equations of incompressible-fluid dynamics in the given coordinate system:

the continuity equation

$$\frac{\partial}{\partial s}(L_\varphi v_s) + \frac{\partial}{\partial n}(L_\varphi L_s v_n) = 0 , \tag{8}$$

the equation of motion in projection onto the s axis

$$\frac{v_s}{L_s}\frac{\partial v_s}{\partial s} + v_n\frac{\partial v_s}{\partial n} + \frac{v_s v_n}{L_s}\frac{\partial L_s}{\partial n} = -\frac{1}{\rho L_s}\frac{\partial p}{\partial s} + \frac{\mu}{\rho L_\varphi}\frac{\partial}{\partial n}\left[\frac{L_\varphi}{L_s}\left(\frac{\partial L_s v_s}{\partial n} - \frac{\partial v_n}{\partial s}\right)\right], \tag{9}$$

the equation of motion in projection onto the n axis

$$\frac{v_s}{L_s}\frac{\partial v_n}{\partial s} + v_n\frac{\partial v_n}{\partial n} - \frac{v_s^2}{L_s}\frac{\partial L_s}{\partial n} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - \frac{\mu}{\rho L_s L_\varphi}\frac{\partial}{\partial s}\left[\frac{L_\varphi}{L_s}\left(\frac{\partial L_s v_s}{\partial n} - \frac{\partial v_n}{\partial s}\right)\right], \tag{10}$$

the energy-transfer equation with neglect of viscous dissipation

$$\rho c\left(\frac{v_s}{L_s}\frac{\partial T}{\partial s}+v_n\frac{\partial T}{\partial n}\right)=\frac{\lambda}{L_s L_\varphi}\left[\frac{\partial}{\partial s}\left(\frac{L_\varphi}{L_s}\frac{\partial T}{\partial s}\right)+\frac{\partial}{\partial n}\left(L_\varphi L_s\frac{\partial T}{\partial n}\right)\right]. \quad (11)$$

To obtain the boundary-layer equations we use the Mises method [4]. As a result, the Lamé coefficients will be equal to $L_n = 1$, $L_s = 1$, and $L_\varphi = r^0$ (in what follows zero is omitted), while Eqs. (8)–(11) are transformed to the following form:

the continuity equation is

$$\frac{\partial r v_s}{\partial s}+\frac{\partial r v_n}{\partial n}=0\,, \quad (12)$$

the equation of motion in projection onto the s axis is

$$\frac{\partial r v_s^2}{\partial s}+\frac{\partial r v_s v_n}{\partial n}=-\frac{r}{\rho}\frac{\partial p}{\partial s}+r\frac{\mu}{\rho}\frac{\partial^2 v_s}{\partial n^2}\,, \quad (13)$$

the equation of motion in projection onto the n axis is

$$\frac{\partial p}{\partial n}=0\,, \quad (14)$$

the energy equation is

$$\frac{\partial r v_s T}{\partial s}+\frac{\partial r v_n T}{\partial n}=\frac{\lambda}{\rho c}r\frac{\partial^2 T}{\partial n^2}\,. \quad (15)$$

In Eq. (13), the pressure gradient is replaced by the velocity derivative outside the boundary layer for which, due to the smallness of the boundary-layer thickness, we take the velocity on the cavity contour in potential flow

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}=V_s\frac{dV_s}{ds}\,.$$

The Squire method [5] is used to determine the heat flux to the cavity wall. Using Eq. (12), we integrate Eqs. (13) and (15) across the boundary layer:

$$\frac{d}{ds}\left(r\int_0^\delta v_s^2 dn\right)-V_s\frac{d}{ds}\left(r\int_0^\delta v_s dn\right)=r\delta V_s\frac{dV_s}{ds}-\frac{\mu}{\rho}r\frac{\partial v_s}{\partial n}\bigg|_{n=0}\,, \quad (16)$$

$$\frac{d}{ds}\left(r\int_0^{\delta_T} v_s\,(T-T_\infty)\right)dn=-\frac{\lambda}{\rho c}r\frac{\partial T}{\partial n}\bigg|_{n=0}\,. \quad (17)$$

Now we assume that the velocity and the temperature are distributed across the boundary layer in the following manner:

$$\frac{v_s}{V_s}=f\left(\frac{n}{\delta}\right),\quad \frac{T-T_\infty}{T_w-T_\infty}=1-f\left(\frac{n}{\delta_T}\right).$$

The heat flux to the wall is equal to

$$q = -\lambda \frac{\partial T}{\partial n}\bigg|_{n=0} = \lambda \frac{T_w - T_\infty}{\delta_T} f'(0) , \quad (18)$$

where the prime denotes differentiation with respect to the argument.

When the relation between the thicknesses δ and δ_T is known, for the heat flux to be determined it is sufficient only to solve Eq. (16) for δ. Work [5] contains the relation $\delta_T = \delta \, \mathrm{Pr}^{-1/3}$, which we will use in calculations. Having substituted the velocity approximation into Eq. (16), we obtain

$$(1 - k_2) \, r\delta \frac{dV_s}{ds} + \frac{d}{ds} [(k_1 - k_2) \, r\delta V_s] = \frac{\mu}{\rho} r \frac{\alpha}{\delta} , \quad (19)$$

where

$$k_1 = \int_0^1 f(x) \, dx ; \quad k_2 = \int_0^1 f^2(x) \, dx ; \quad \alpha = f'(0) .$$

If the form parameters k_1, k_2, and α are constant, the solution of Eq. (19) will be

$$\delta^2 = \frac{2\alpha}{k_1 - k_2} \frac{\mu}{\rho} \frac{1}{r^2 V_s^{2+m}} \int_0^s r^2 V_s^{1+m} \, ds , \quad m = 2 \frac{1 - k_2}{k_1 - k_2} . \quad (20)$$

Since the cavity contour is a streamline, on the contour we have

$$V_s^2 = V_r^2 + V_z^2 , \quad \frac{ds}{V_s} = \frac{dr}{V_r} = \frac{dz}{V_z} .$$

Using the determinations of the velocities (5) and the contour equation (6), substituting the solution (20) into Eq. (19) and then into Eq. (18), and determining the constants a and A, as has been indicated above, we obtain the expression for the local Nusselt number:

$$\frac{\mathrm{Nu}}{\mathrm{Re}^{1/2} \, \mathrm{Pr}^{1/3}} = \frac{q d_c}{\lambda (T_w - T_\infty)} = \sqrt{\frac{(k_1 - k_2) \, \alpha \eta_1^{(1)}}{[\exp(ah) - 1] \, S(\eta)}} ; \quad \mathrm{Re} = \frac{\rho V_0 d_c}{\mu} , \quad (21)$$

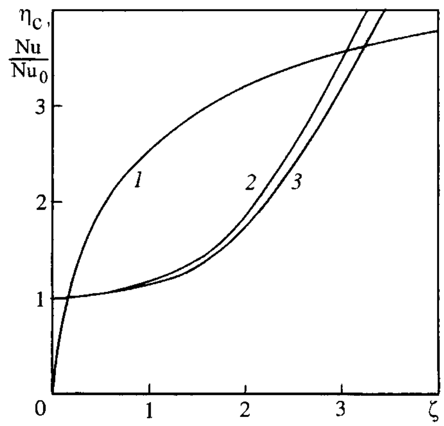
where

$$d_c = 2R_c ; \quad S(\eta) = \frac{2 \int_0^\eta x \Psi(x) \, dx}{\eta^2 \Psi(\eta)} ; \quad \Psi(\eta) = \left[\left(1 - \frac{\eta}{2} \frac{J_0(\eta)}{J_1(\eta)} \right)^2 + \frac{\eta^2}{4} \right]^{\frac{2+m}{2}} .$$

At the critical point, we have

$$S = S_0 = \frac{2}{4 + m} , \quad \mathrm{Nu} = \mathrm{Nu}_0 = \sqrt{\frac{(1 + 2k_1 - 3k_2) \, \alpha \eta_1^{(1)}}{\exp(ah) - 1}} \, \mathrm{Re}^{1/2} \, \mathrm{Pr}^{1/3} . \quad (22)$$

We dwell on the solution when the velocity profile is assigned by the von Kármán–Pohlhausen formula [5]



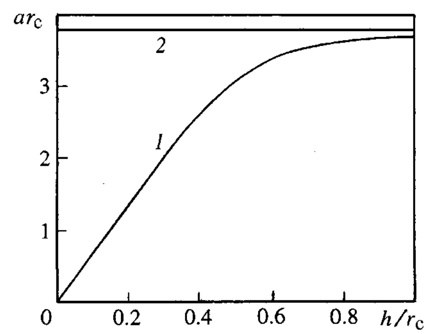


Fig. 3. Cavity contour η_c and the ratio of the Nusselt numbers at the given point of the contour and at the stagnation point: 1) η_c; 2) Nu/Nu_0 for the von Kármán–Pohlhausen velocity profile; 3) Nu/Nu_0 for the profile $f(x) = x$.

Fig. 4. Plot for determination of the parameter a (curve 1) and $ar_c = \eta_1^{(1)}$ (curve 2).

$$f(x) = F(x) + \Lambda G(x) , \quad x = \frac{n}{\delta} , \quad \Lambda = \frac{\delta^2 V_s' \rho}{\mu} , \quad F(x) = 2x - 2x^3 + x^4 ; \quad G(x) = \frac{1}{6}(x - 3x^2 + 3x^3 - x^4) .$$

Whence we determine the form parameters of the profile

$$1 - k_2 = \frac{263}{630} - \frac{71}{7560}\Lambda - \frac{\Lambda^2}{9072} , \quad k_1 - k_2 = \frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072} , \quad \alpha = 2 + \frac{\Lambda}{6} .$$

We write Eq. (19) in terms of the parameter Λ:

$$\frac{d}{ds}\left[\Lambda (k_1 - k_2)^2 \frac{r^2}{V_s'}\right] = 2(k_1 - k_2)\frac{r^2}{V_s}[\alpha - \Lambda(1 - k_1) - \Lambda(k_1 - k_2)] .$$

From this equation it follows that in the neighborhood of the critical point, where $V_s = \beta s$ and $s \approx r$, we have the relation

$$\Lambda(k_1 - k_2) = \alpha - \Lambda(1 - k_2) - \Lambda(k_1 - k_2) .$$

which can be fulfilled on condition that $\Lambda = 4.716$ (another solution $\Lambda = 21.142$ gives a physically incorrect velocity profile). The determinations of Λ and Nu yield

$$\frac{\delta}{d_c} = \frac{1}{\sqrt{\mathrm{Re}}}\sqrt{\frac{\Lambda}{(\bar{V}_s)_\xi'}\frac{\exp(ah) - 1}{2\eta_1^{(1)}}} , \quad \frac{\mathrm{Nu}}{\mathrm{Pr}^{1/3}\,\mathrm{Re}^{1/2}} = \alpha\sqrt{\frac{(\bar{V}_s)_\xi'}{\Lambda}\frac{2\eta_1^{(1)}}{\exp(ah) - 1}} , \quad \xi = as . \tag{23}$$

In Fig. 3, along with the form of the cavity contour, we present the results of calculations of the ratio Nu/Nu_0 as a function of the dimensionless coordinate ζ for the von Kármán–Pohlhausen profile (curve 2) and for the simplest approximation of the velocity profile $f(x) = x$ (curve 3). As is seen, the difference in the calculated heat fluxes is not large. It should also be noted that curve 2 is no different, in practice, from the curve calculated for the profile:

$$f(x) = \frac{3}{2}x - \frac{1}{2}x^3 .$$

The increase in heat transfer with distance from the critical point is attributed to the exponential growth in the velocity along the contour, which results in a decrease in the boundary-layer thickness in this direction and not in an increase (the ratio Nu/Nu_0 is in inverse proportion to the ratio δ/δ_0).

In practical application of this theory to calculation of the heat transfer to the cavity walls, the question arises of whether it is possible to determine the parameter a if the real dimensions of the cavity radius $r_c (r_c < R_c)$ and of the depth h are known. For the cavity whose form is described by expression (6) we find

$$\frac{h}{r_c} = \frac{1}{\eta} \ln \left(\frac{\eta}{2J_1(\eta)} \right).$$

Setting $\eta = ar_c$, from this transcendental equation we can find the parameter a as a function of h/r_c (Fig. 4). It is evident that already for h/r_c of the order of 1 the quantity r_c can be considered to be equal, in practice, to the limiting R_c. For a small cavity depth, $a = 8h/r_c^2$.

Attention should be paid to the exponential influence of the cavity depth on the Nusselt number in formulas (21)–(23). For a rather large ratio h/r_c, the heat flux to the walls decreases sharply. Probably, that is the reason why the efficiency of the jet methods of drilling of deep wells is low.

NOTATION

a, coefficient in Eq. (4), 1/m; A and B, constants in Eq. (4), m/sec; r, φ, z, cylindrical coordinates; n, s, φ, curvilinear coordinates; r^0 and z^0, coordinates of the point ($n = 0$, s, φ) in the cylindrical coordinate system; $x = n/\delta$, dimensionless coordinate; $\zeta = s/\delta$, dimensionless coordinate; V_r and V_z, radial and axial velocities of potential flow, m/sec; v_r and v_z, radial and axial velocities in the boundary layer, m/sec; v_n and v_s, velocity components in the curvilinear coordinate system, m/sec; V_0, axial velocity at the assigned point on the cavity axis, m/sec; R, radius of curvature of the cavity contour, m; R_c, limiting radius of the cavity, m; d_c, limiting diameter of the cavity, m; r_c, running radius of the cavity, m; dl, element of length in the curvilinear coordinate system, m; L_s, L_n, and L_φ, Lamé coefficients; h, cavity depth, m; T, temperature, K; p, pressure, N/m^2; c, heat capacity, J/(kg·K); q, heat flux, W/m^2; ρ, density, kg/m^3; λ, coefficient of thermal conductivity, W/(m·K); μ, coefficient of viscosity, kg/(m·sec); δ, thickness of the dynamic boundary layer, m; δ_T, thickness of the thermal boundary layer, m; $J_1(\eta)$, Bessel function of first order; $\eta_1^{(1)}$, first root of the function $J_1(\eta)$; ψ, stream function, m^2/sec; $f_0(z)$, velocity distribution along the z axis; m, exponent in Eq. (20); $S(\eta)$, function in Eq. (21); $\Psi(\eta)$, function in determination of $S(\eta)$; Λ, dimensionless number in the formula of the von Kármán–Pohlhausen profile; $f(n/\delta)$ and $f(n/\delta_T)$, dimensionless velocity and temperature distributions in the boundary layer; η, dimensionless coordinate; $P(ar)$, function entering into the determination of the stream function; β, velocity gradient in the neighborhood of the critical point; Pr, Prandtl number; Re, Reynolds number; Nu, Nusselt number. Subscripts: 0, critical point; c, cavity; w, wall; ∞, edge of the boundary layer; 1 and 2, numbers of the form parameters of the boundary layer.

REFERENCES

1. L. G. Loitsyanskii, *Mechanics of Liquids and Gases* [in Russian], Moscow (1957).
2. E. F. Avdeev and S. L. Dorokhovich, in: *Proc. 2nd Russ. Nat. Conf. on Heat Transfer* [in Russian], Vol. 2, Moscow (1998), pp. 33–36.
3. I. Yu. Linenko-Mel'nikov and B. A. Uryukov, *Izv. Sib. Otd. Akad. Nauk SSSR*, *Ser. Tekh. Nauk*, No. 10, Issue 3, 74–85 (1985).
4. N. E. Kochin, I. E. Kibel', and N. V. Roze, *Theoretical Hydromechanics* [in Russian], Vol. 2, Moscow–Leningrad (1948).
5. H. Schlichting, *Boundary-Layer Theory* [Russian translation], Moscow (1977).